THE METHOD OF IMAGES FOR SOLVING THE EQUATIONS OF HEAT CONDUCTION IN LAYERED MEDIA

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One obtains the temperature field generated by an instantaneous point source in the layer. It is shown that the solution can be obtained by the method of images in the plane.

In the consideration of heat-transfer processes in radioelectronic systems, one encounters problems of determination of the temperature field generated by an instantaneous point source of heat in a layer which is in contact with a rigid medium (Fig. 1). The media have different thermal coefficients. Such a problem has not been considered previously. Only the solutions of the heat conduction equations for a layer with homogeneous boundary conditions are known [1, 2]. The case of two adjoining half spaces with different thermal coefficients is examined in [3].

In order to find the temperature field in the layer one solves the system of differential equations

$$\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T_1}{\partial z^2}-\frac{1}{a_1^2}\frac{\partial T_1}{\partial t}=\frac{-q(r,\ z,\ t)}{\lambda_1},$$

$$\frac{\partial^2 T_2}{\partial r^2}+\frac{1}{r}\frac{\partial T_2}{\partial r}+\frac{\partial^2 T_2}{\partial z^2}-\frac{1}{a_2^2}\frac{\partial T_2}{\partial t}=0,$$

$$\frac{\partial T_1}{\partial z}=0,\quad z=d, \tag{1}$$

$$\left.\begin{aligned} T_1&=T_2\\ \lambda_1\frac{\partial T_1}{\partial z}&=\lambda_2\frac{\partial T_2}{\partial z}\end{aligned}\right\}\quad (z=0),$$

$$q(r,\ z,\ t)=\begin{cases} q, & r=0;\quad t=0;\quad z=z',\\ 0, & t>0.\end{cases}$$

The system (1) in terms of the images is satisfied by the functions:

$$\overline{T}_1=\frac{q}{4\pi\lambda_1}\int\limits_0^\infty\frac{\xi}{\eta_1}J_0(\xi r)\{\exp[-\eta_1|z-z'|]+A\exp[\eta_1 z]+B\exp[-\eta_1 z]\}d\xi; \tag{2}$$

$$\overline{T}_2=\frac{q}{4\pi\lambda_1}\int\limits_0^\infty\frac{\xi}{\eta_2}J_0(\xi r)\,C\exp[\eta_2 z]\,d\xi. \tag{3}$$

The first term in (2) represents the Green function for an instantaneous source in the space. The constants A, B, and C are obtained from the boundary conditions:

$$A=De^{-2\eta_1 d};\quad B=D-e^{\eta_1 z'};\quad C=\frac{2\lambda_1\eta_1}{\lambda_1\eta_1+\lambda_2\eta_2}\left(De^{-2\eta_1 d}+e^{-\eta_1 z'}\right);\quad D=\frac{\gamma e^{-\eta_1 z'}+e^{\eta_1 z'}}{1-\gamma e^{-2\eta_1 d}}.$$

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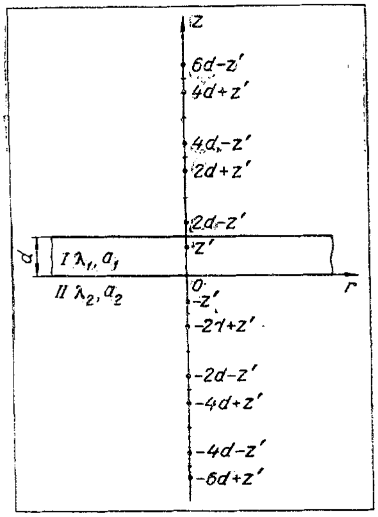

Fig. 1. The representation of the sources in the planes z = 0 and z = d.

We represent the denominator of D in the form

$$\frac{1}{1-\gamma e^{-2\eta_1 d}}=\sum_{n=0}^{\infty}\gamma^n e^{-2n\eta_1 d},$$

and γ^n as the binomial

$$\left(1-\frac{2\lambda_2\eta_2}{\lambda_1\eta_1+\lambda_2\eta_2}\right)^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^k}.$$

Then (2) and (3) can be rewritten in the form

$$\bar{T}_1=\frac{q}{4\pi\lambda_1}\int_0^{\infty}\frac{\xi J_0(\xi r)}{\eta_1}\Big\{\exp[-\eta_1|z-z'|]+\exp[-\eta_1(2d-z-z')]$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^k}\left[\exp[-\eta_1(2d+z+z')]+\exp[-\eta_1(2nd+2d-z+z')]\right]$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^k}\left[\exp[-\eta_1(2nd+z-z')]+\exp[-\eta_1(2nd+2d-z-z')]\right]\Big\}d\xi; \tag{4}$$

$$\bar{T}_2=\frac{q}{2\pi}\int_0^{\infty}\xi J_0(\xi r)\Big\{\frac{\exp[-\eta_1 z']}{\lambda_1\eta_1+\lambda_2\eta_2}$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^{k+1}}\exp\left[-\eta_1[2(n+1)d+z']\right]$$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^{k+1}}\exp\left[-\eta_1[2(n+1)d-z']\right]\Big\}\exp[\eta_2 z]\,d\xi. \tag{5}$$

The expressions (4) and (5) contain integrals of the form

$$\bar{I}_{10}(z)=\int_0^{\infty}\xi J_0(\xi r)\frac{\exp[-\eta_1 z]}{\eta_1}d\xi;$$

$$\overline{I}_{1k}(z)=\int_0^\infty \xi J_0(\xi r)\frac{(2\lambda_2\eta_2)^k}{(\lambda_1\eta_1+\lambda_2\eta_2)^k}\exp[-\eta_1 z]\,d\xi;$$

$$\overline{I}_{2k}(z_1)=\int_0^\infty \xi J_0(\xi r)\frac{(2\lambda_2\eta_2)^{k-1}}{(\lambda_1\eta_1+\lambda_2\eta_2)^k}\exp[-\eta_1 z_1+\eta_2 z]\,d\xi.$$

The first integral gives the temperature field generated by an instantaneous point source in the unbounded space and can be easily transformed to the preimage

$$I_{10}=\frac{\exp\left[-\dfrac{r^2+z^2}{4a_1^2 t}\right]}{2a_1\sqrt{\pi}\,t^{3/2}}.$$

For the computation of the integrals $\overline{I}_{1k}$ and $\overline{I}_{2k}$ we eliminate the expression $(\lambda_1\eta_1+\lambda_2\eta_2)^k$ from the denominators by making use of the equality [4]:

$$\int_0^\infty \zeta^k\exp[-\mu\zeta]\,d\zeta=k!\,\mu^{-k-1}.$$

Then

$$\overline{I}_{1k}=\frac{2^k\lambda_2^k}{(k-1)!}\int_0^\infty \xi J_0(\xi r)\,d\xi\int_0^\infty \frac{\eta_2^k}{\eta_1}\zeta^{k-1}\exp[-(\lambda_1\eta_1+\lambda_2\eta_2)\zeta-\eta_1 z]\,d\zeta.$$

We go back to the preimage. Assuming that the order of integration may be interchanged and making use of the convolution theorem, we obtain

$$I_{1k}=\frac{a_1}{2\pi(k-1)!}\left(\frac{\lambda_2}{a_2}\right)^k\int_0^\infty \xi J_0(\xi r)\,d\xi\int_0^\infty \zeta^{k-1}d\zeta$$

$$\times\int_0^t \frac{\mathrm{He}_{k+1}\left(\dfrac{\lambda_2\zeta+z}{2a_2\sqrt{\tau}}\right)}{\tau^{\frac{k+2}{2}}(t-\tau)^{\frac{1}{2}}}\exp\left\{-\frac{\lambda_2^2\zeta^2}{4a_2^2\tau}-\frac{(\lambda_1\zeta+z)^2}{4a_1^2(t-\tau)}-a_2^2\xi^2\tau-a_1^2\xi^2(t-\tau)\right\}d\tau.$$

The ξ-integral is evaluated by making use of the relation [2]

$$\int_0^\infty \xi J_0(\xi r)e^{-b\xi^2}\,d\xi=\frac{1}{2b}\exp\left[-\frac{r^2}{4b}\right].$$

In order to evaluate the ζ-integral, we write the Hermite polynomials He_k in the form [5]

$$\mathrm{He}_{k+1}\left(\frac{\lambda_2\zeta+z}{2a_2\sqrt{\tau}}\right)=(k+1)!\sum_{m=0}^{\left[\frac{k+1}{2}\right]}(-1)^m\frac{1}{m!\,(k+1-2m)!}\left(\frac{\lambda_2\zeta}{a_2\sqrt{\tau}}\right)^{k-2m+1}$$

Then, the ζ-integral reduces to the evaluation of the integral [4]

$$\int_0^\infty \zeta^{k-1}\exp[-\beta\zeta^2-\gamma\zeta]\,d\zeta=\frac{\Gamma(k)}{(2\beta)^{k/2}}\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-k}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$

and the integral I_{1k} takes the form

$$I_{1k}(z)=\frac{k(k+1)}{4\pi}a_1\sum_{m=0}^{\left[\frac{k+1}{2}\right]}(-1)^m\left(\frac{\lambda_2}{a_2}\right)^{2(k-m)+1}\frac{\Gamma(2k-2m+1)}{m!\,(k-2m+1)!}$$

$$\times\int_0^t\frac{\exp\left\{-\dfrac{r^2}{4[a_1^2(t-\tau)+a_2^2\tau]}-\dfrac{z^2}{4a_1^2(t-\tau)}+\dfrac{\delta_1^2}{8\beta_1}\right\}}{(2\beta_1)^{\frac{2(k-m)}{2}}(t-\tau)^{\frac{1}{2}}\tau^{\frac{2(k-m+1)+1}{2}}\cdot[a_1^2(t-\tau)+a_2^2\tau]}D_{-2(k-m)-1}\left(\frac{\delta_1}{\sqrt{2\beta_1}}\right)d\tau. \qquad (6)$$

The integral $\overline{I}_{2k}$ is evaluated similarly

$$I_{2k}(z_1) = \frac{k}{8\pi a_1}\left(\frac{\lambda_2}{a_2}\right)^{k-1}\sum_{m=0}^{\left[\frac{k}{2}\right]}\sum_{l=0}^{k-2m}(-1)^{k-m-l}\frac{z^{k-2m-l}}{m!\,(k-2m)!}\frac{\lambda_2^l}{a_2^{k-2m}}\binom{k-2m}{l}$$
$$\times\int_0^t \frac{\exp\left\{-\frac{r^2}{4[a_1^2(t-\tau)+a_2^2\tau]}-\frac{z^2}{4a_2^2\tau}-\frac{z_1^2}{4a_1^2(t-\tau)}+\frac{\delta_2^2}{8\beta_2}\right\}}{(t-\tau)^{3/2}\,\tau^{\frac{2(k-m)+1}{2}}\,[a_1^2(t-\tau)+a_2^2\tau]\,(2\beta_2)^{\frac{k+l}{2}}}$$
$$\times\left[\frac{\lambda_1\Gamma(k+l+1)}{\sqrt{2\beta_2}}D_{-k-l-1}\left(\frac{\delta_2}{\sqrt{2\beta_2}}\right)+z_1\Gamma(k+l)\,D_{-k-l}\left(\frac{\delta_2}{\sqrt{2\beta_2}}\right)\right]d\tau. \tag{7}$$

$\overline{I}_{1k}(z)$ and $\overline{I}_{2k}(z_1)$ have under the integral sign elementary functions and the parabolic cylinder function $D_{-k-1}(z)$, which can be expressed in terms of the error function [5]

$$D_{-k-1}(z) = \sqrt{\frac{\pi}{2}}\frac{(-1)^k}{k!}e^{-\frac{z^2}{4}}\frac{d^k}{dz^k}\left\{e^{\frac{z^2}{2}}\left[1-\Phi\left(\frac{z}{\sqrt{2}}\right)\right]\right\},$$

and can be computed on a digital computer.

Finally, the temperature fields in the media 1 and 2 are described by the expressions:

$$T_1 = \frac{q}{4\pi\lambda_1}\left\{I_{10}(z-z') + I_{10}(2d-z-z')\right.$$
$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}[I_{1k}(-2nd-z-z')+I_{1k}(2nd+2d-z+z')]$$
$$\left.+\sum_{n=1}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}[I_{1k}(-2nd-z+z')+I_{1k}(2nd+2d-z-z')]\right\}, \tag{8}$$

$$T_2 = \frac{q}{2\pi}\left\{I_{21}(z')+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}I_{2(k+1)}[2(n+1)d+z']\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}I_{2(k+1)}[2(n+1)d-z']\right\}. \tag{9}$$

The solution of the problem can be obtained by the method of images in the plane, if we assume that by the reflection of the sources in the plane z = 0 there appears the additional term $I_{1k}(z)$. Indeed, the first term in (8) describes the temperature field generated by the instantaneous point source in the unbounded space, the second term describes the field generated by the reflection of the first point in the adiabatic plane z = d. The reflection of these sources in the plane z = 0 leads to the appearance at the points (0, −z′) and (0, −2d + z′) of instantaneous point sources, corresponding to the terms $I_{10}(z - z')$ and $I_{10}(2d + z + z')$ and of additional sources, corresponding to $I_{11}(z - z')$ and $I_{11}(2d + z + z')$. Each of the following reflections in the plane z = 0 leads to the appearance of an additional source. Thus, at the points (0, −4d − z′) and (0, −6d + z′), corresponding to the third reflection, there are four sources available (Fig. 1).

The expression (8) can be easily transformed into the solution of the problem with the adiabatic plane z = 0 or with zero temperature on this plane. We consider the expression (4). In the first case $\lambda_2 = 0$, the finite sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2\lambda_2\eta_2}{\lambda_1\eta_1+\lambda_2\eta_2}\right)^k = 1,$$

and in (8) only series of the form $\sum_{n=0}^{\infty} I_{10}(z)$ are left. In the second case $\lambda_2 \to \infty$, the terms corresponding to $I_{1k}(z)$ tend to two and in (8) there are left series of the form

$$\sum_{n=0}^{\infty}(-1)^n I_{10}(z).$$

NOTATION

T	is the temperature;
λ_1, λ_2, a_1, a_2	are the coefficients of thermal conductivity and thermal diffusivity in the media 1 and 2;
q(r, z, t)	is the density of the heat source;

$\eta_1 = \sqrt{\frac{p}{a_1^2} + \xi^2}$;

$\eta_2 = \sqrt{\frac{p}{a_2^2} + \xi^2}$;

$J_0(\xi r)$	is the Bessel function of the first kind and zero order;

$\gamma = (\lambda_1\eta_1 - \lambda_2\eta_2)/(\lambda_1\eta_1 + \lambda_2\eta_2)$;

$\binom{n}{k} = \frac{n(n-1)\ \ldots\ (n-k+1)}{1 \cdot 2\ \ldots\ k}, \quad \binom{n}{0} = 1$;

$He_k(z)$	are the Hermite polynomials;
$\Gamma(k) = (k-1)!$	is the γ-function;
$\left[\frac{k+1}{2}\right] = \frac{k+1}{2}$	if k is odd;
$\left[\frac{k+1}{2}\right] = k/2,$	if k is even;
$\Phi(z)$	is the error function;
$D_{-k}(z)$	is the parabolic cylinder function;

$\delta_1 = \frac{\lambda_1 z}{2a_1^2\tau}$;

$\delta_2 = \frac{1}{2}\left[\frac{\lambda_1 z}{a_1^2(t-\tau)} - \frac{\lambda_2 z}{a_2^2\tau}\right]$;

$\beta_1 = \frac{1}{4}\left[\frac{\lambda_1^2}{a_1^2\tau} + \frac{\lambda_2^2}{a_2^2(t-\tau)}\right]$;

$\beta_2 = \frac{1}{4}\left[\frac{\lambda_1^2}{a_1^2(t-\tau)} + \frac{\lambda_2^2}{a_2^2\tau}\right]$.

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